Maria Teresa Borgato (Ferrara, Italy)

CONTINUITY AND DISCONTINUITY IN ITALIAN MATHEMATICS AFTER THE UNIFICATION: FROM BRIOSCHI TO PEANO

A period of renewal for Italian mathematics started with the Italian Risorgimento in the middle of the 19th century. A previous one had occurred at the end of the 18th century during the Napoleonic period, when a partial unification of Italy took place. In the first half of the 19th century Italian mathematicians were still essentially linked to French polytechnic models: among them were Ottaviano Fabrizio Mossotti, Antonio Bordoni, Giorgio Bidone, and Giovanni Plana. These models had undergone a crisis in France, too, following the development of mathematical research in the German universities which had opened up new horizons (Steiner, Jacobi, and Möbius). The young Italian mathematicians who measured themselves with the highest scientific level understood that they had to turn to the German schools, above all those of Berlin and Göttingen[1].

Elements of continuity can also be pointed out, but in this case the main element of novelty with respect to the previous situation was a deep change in the direction of mathematical research with the new reference point of the German schools and their emerging ideas and concepts, which can better explain the birth and the progress of a new school of Italian mathematicians at an international level[2].

1. Political unification. Renewal of scientific research and education

As the Napoleonic period drew to a close, in Italy intellectual energies which had been released, producing both a civic and scientific renewal now were forced to retire into themselves. Many scientists who had collaborated with the Jacobean Republics or the Napoleon government were forced into exile, an example being Ottaviano Fabrizio Mossotti who, in 1823, emigrated first to London and then to Argentina[3].

In the period prior to the Unification there was, however, no lack of good scholars of mathematical sciences in Italy, like Plana in Turin, Paoli and

---

[1] *Abregé d'histoire des mathématiques 1700–1900*, (ed.) J. Dieudonné, 2 vols, Hermann, Paris 1976, *La matematica in Italia (1800–1950)*, (eds) E. Giusti & L. Pepe, Polistampa, Firenze 2001.

[2] D. J. Struik, *Matematica: un profilo storico* with an appendix by U. Bottazzini, Il Mulino, Bologna 1981.

[3] *Universitari italiani nel Risorgimento*, (ed.) L. Pepe, Clueb, Bologna 2002.

Mossotti in Pisa, Bellavitis in Padua, Fergola in Naples, Ruffini in Modena, and Bordoni in Pavia, but political upheaval and division had resulted in a lack of common scientific activity and of efficient organisation at university level. It was not easy to disseminate the results of new scientific research in spite of several scientific congresses held in Italy between 1839 and 1847. The year 1848 brought about a new period in Italy and in Europe, as well as a new generation of scientists who took an active part in both the political and cultural activity of the Italian *Risorgimento*. At the forefront we find Francesco Brioschi, Enrico Betti and Luigi Cremona[1].

Francesco Brioschi (1824–1897), a pupil of Antonio Bordoni at the University of Pavia, where he graduated in engineering in 1845, was politically active within the Lombard patriotic movement inspired by Giuseppe Mazzini. He took part in the insurrection of the Five Days of Milan against the Austrian government (18–22 March 1848), one of the most famous events of the First War of Independence which led to the liberation of Lombardy. He was imprisoned on the first day of the rising and was freed by the insurgents. After the Austrian restoration he continued to participate in the Lombard resistance and in 1849 he was a member of the Central Committee of Milan. Enrico Betti (1823–1892), together with his professor Ottaviano Fabrizio Mossotti and other students and professors of the University of Pisa, took part in the battle of Curtatone and Montanara (29th May 1848), where a division composed of battalions of Tuscan universities stopped the advance of the Austrian army. As a member of the battalion of Pisa there was also the future physicist Riccardo Felici, a disciple of Mossotti and Carlo Matteucci who studied electro–magnetic phenomena. Luigi Cremona (1830–1903) was among the 160 voluntary Neapolitan students who in 1848 arrived in Pavia to support the temporary government of Milan, which had been set up against the Austrians. He displayed heroic qualities in the defence of the Venetian Republic. On his return from the war he enrolled in the University of Pavia to study under the guidance of Bordoni and Brioschi. In general, the best Italian mathematicians and scientists of that period took an active part in the fight for the Italian independence and, with the constitution of the united state, assumed important government posts. The years 1858–1860 constituted a crucial period for political activity and scientific renewal in Italy: between 26th April 1859 and 12th July 1859 the Second War of Independence was fought between the French–Piemontese and Imperial Austrian armies, the outcome of which led to a partial unification by the annexation of Lombardy to the Kingdom of Sardinia laying the foundations for the constitution of the Kingdom of Italy.

The same years witnessed other significant events of sweeping changes in the development and direction of Italian mathematics. From 1858 on there was the publication in Italy of the first journal devoted exclusively to mathematical

[1] Their collected papers in national editions: *Opere matematiche di Francesco Brioschi*, Hoepli, Milano 1901–1909, 5 vols, *Opere matematiche di Enrico Betti*, R. Accademia de' Lincei, Hoepli, Milano 1903–1913, 2 vols, *Opere matematiche di Luigi Cremona*, R. Accademia de' Lincei, Milano, Hoepli, 1914–1917, 3 vols. The edition of their scientific correspondence found in the Archives and Italian Libraries has proved to be a great source for the reconstruction of that glorious period. See: *Francesco Brioschi e il suo tempo (1824–1897),* (eds.) C. G. Lacaita & A. Silvestri, Franco Angeli, Milano 2000–2003, 3 vols.

research: the *Annali di Matematica Pura ed Applicata* substituted the previous journal called the *Annali di Scienze Matematiche e Fisiche*. The title of the new journal took its inspiration from Liouville's famous journal, but was also modelled on *Crelle's Journal*. The editorial board was composed of Francesco Brioschi, Enrico Betti, Barnaba Tortolini and Angelo Genocchi. It became an important vehicle for the dissemination abroad of Italian mathematical research as well as providing Italians with an update on the development of research from other countries. The *Annali* were initially edited by Tortolini at the headquarters in Rome, but, in 1867, were transferred to Milan where Brioschi took over as editor, at first in collaboration with Cremona, and then alone. In the following thirty years until Brioschi's death other 26 volumes of the *Annali* were published with contributions from both Italian and European scientists such as Lamé, Jacobi, Kronecker, Raabe, Hermite, Cayley, Weierstrass, Riemann, and Sylvester, witness to the stimulation and development which Italian mathematical research underwent in that period.

The same year, 1858, Francesco Brioschi, Enrico Betti and Felice Casorati undertook a journey to visit the leading European research centres, and meet up with the most important mathematicians in order to discuss new lines of thought, and establish contacts for further collaboration with the new journal. The journey also provided an opportunity to compare organisation methods for research and higher education. Cremona and Placido Tardy, a mathematician from Genoa and friend of Betti, Brioschi and Genocchi, were unable to take part in the journey. Brioschi got his young assistant, Felice Casorati, to accompany him. The journey lasted less than two months, from 20th September to 1st November 1858, during which time they met almost all of the leading French and German mathematicians. In Berlin they spent many hours with Borchardt, Kronecker, Kummer and Weierstrass and also met Aronhold and Schellbach. In Göttingen they met Riemann, Stern, Dedekind, in Heidelberg, Hesse and Cantor, in Karlsrhue, Dienger and Clebsch, in Leipzig, Möbius, in Dresden, Baltzer and Schlömilch, in Paris they met Poncelet, Bertrand, Duhamel, Steiner, Hermite, Chasles, Terquem, Lebesgue, Prohuet and Bonnet. The only people they were not able to meet were Dirichlet in Göttingen and Liouville in Paris. Further meetings took place in the following years between Betti and Riemann and between Casorati and Weierstrass.

2. Changes in the direction of mathematical research in Italy

Brioschi and Betti were well known abroad for the important results they had obtained in several avant–garde research fields of mathematics. They had managed to break free from the isolation felt by many other Italian scientists. The journey and contact with German mathematicians were to bring about a great change particularly in the direction of their future research and that of their pupils and more generally on Italian mathematics with the circulation of the *Annali*. Let us examine this evolution of themes and methods in the individuals it involved.

Before their journey Betti and Brioschi's point of reference was mainly the French and English schools. Brioschi had been a professor at the

University of Pavia since 1850 and had developed the new theory of determinants and the theory of forms of two or more variables, referring to the studies of Cayley, Jacobi, Hermite and Sylvester. The field in which he was more deeply interested and in which he obtained more success was that of algebraic equations and the new related algorithms, invariants, covariants, algebraic and symbolic forms. His most important result was the solution of the fifth degree equations which he shared with Hermite and Kronecker. Consequently, at the end of the 1850s, Brioschi investigated the closely linked theory of elliptic and Abelian functions, to which he contributed new results.

Betti had just returned to Pisa, where he was charged with a teaching course at the University. His scientific interests, in the 1850s, which he had developed through his contact with his professor, Mossotti, and with Sylvester, concerned the theories of algebraic equations and invariants. Betti had provided the general conditions for solvability by radicals and extended Kronecker's results on prime degree equations and had demonstrated, for the first time, theorems formulated by Evariste Galois in 1832. From 1853 Betti worked on the analytic solution of the fifth degree equation by means of elliptic functions and applied the Galois theory to the reduction of degree of modular equations[1]. Then he focused on invariant theory and on the elimination theory for symmetric functions of the common roots of two equations.

Of the three mathematicians, Brioschi appears to be the one who remained most faithful to problems and methods that had characterised his first research works. However, also in Brioschi we find links to the German authors, mainly to the school of Weierstrass, which are particularly evident in the field of the Abelian functions in the post unification period. Let us analyse some of his works written around the year 1858. Brioschi published a work on the division of hyperelliptic integrals, closely linked to Hermite's research on the theory of the transformation of the Abelian functions in which he determined the coefficients of homogeneous polynomials which provide these transformations[2]. However, in the same year 1858, Brioschi published a memoir on the Abelian functions which develops some parts of the theory that Weierstrass had presented in his celebrated memoir of 1856, the last one that Weierstrass wrote on this topic. In the years 1863–1864 in some of his memoirs strictly linked to those of Aronhold, following the line of Weierstrass' ideas, he accomplished an algebraic reduction of the Abelian integral, in the case of a cubic equation between the variables, based on the theory of covariants of the ternary forms. Brioschi proceeded with his research on the theory of covariants and invariants of binary forms, which were themes developed by the English and French schools, using, however, the symbolic representation which had been introduced by Aronhold and Clebsch. He continued his studies on the resolution of algebraic equations and reached the solution of the sixth degree equation by means of the hyperelliptic functions.

[1] I. Nagliati, *Le prime ricerche di Enrico Betti nel carteggio con Mossotti* in: *Bollettino di storia delle scienze matematiche* 20, 2000, pp. 3–86.

[2] F. Brioschi, *Sur la théorie de la transformation des fonctions abéliennes* in: *Comptes Rendus* 47, 1858, pp. 310–313.

Luigi Cremona (1830–1903) is considered the founder of the Italian school of geometry. In Europe a renewal of geometry spread from France to Europe at the beginning of the century, with the teaching of Monge and Carnot at the Ecole Polytechnique and the pupils of that school: Brianchon, Poncelet, and Chasles. But no Italian had taken part in this. The new methods of projective geometry allowed a general and holistic treatment of a whole class of problems that until then only analytic methods had permitted. Of the Italian mathematicians of that period, Cremona was the one devoted to pure geometry and a chair of Higher Geometry was created for him at the University of Bologna (1860). Initially his reference was the French school and Michel Chasles in particular to whom he gave the credit of having oriented him to the study of pure geometry after a formation which was exclusively based on analytic methods. Later his research took a different direction after he had studied the German mathematicians, Steiner in particular.

His scientific production is made up of different phases: from his graduation in engineering and architecture (1853) to the chair of higher geometry in Bologna (1860) he published memoirs on geometry which were inspired by the themes and analytic methods of his mentors Bordoni and Brioschi: on the conjugate tangents in contact points of surfaces with the envelops**,** properties of the conics and surfaces of second order, space curves. From 1861 to 1864 he once again took up the theme of space cubics with purely synthetic methods and he developed new research on third degree ruled surfaces. Influenced by Steiner's memoir on general properties of algebraic curves[1] which only contained results, but without demonstrations, Cremona published the general theory of algebraic curves from a synthetic point of view, starting from the theory of polar curves according to Grassmann's construction (1862), followed by a second treatise devoted to the geometric theory of surfaces (1866). The same year he received an award from the Academy of Berlin for his memoir on third order surfaces in which he gave a synthetic demonstration of properties enunciated without demonstration by Steiner. His most famous research works were those on birational transformations, namely the rational biunivocal correspondences between surfaces which were named after him and which were generalisations of a concept that derived from Riemann's Abelian functions. Cremona was famed throughout Europe and the theory of Cremonian transformations found frequent application to the theory of algebraic curves and surfaces and in systems of plane curves[2].

3. The influence of Riemann

Betti was greatly influenced by Riemann who had informed him of his new ideas on topology. When Riemann came to stay in Pisa in 1863, contact between the two was renewed on almost a daily basis. For reasons of health,

[1] J. Steiner, *Allgemeine Eigenschaften der algebraischen Curven* in: *Crelle's Journal* 47, 1854, pp. 1–6.

[2] *Un secolo di progresso scientifico italiano 1839–1939*, S.I.P.S., Roma 1939–1940, 7 vols .

Riemann, who died at the age of forty, spent most of the last four years of his life in Italy[1].

In 1859 Betti's translation into Italian of Riemann's doctoral thesis submitted in 1851 was inserted into the *Annali di Matematica* as: *Fondamenti di una teorica generale delle funzioni di una variabile complessa*. Between 1860 and 1861 Betti published in the *Annali* his own theory on elliptic functions, inspired by the *Theorie der Abel'schen Funktionen* published by Riemann in 1857, in which Riemann resolved the general problem of the inversion of Abelian integrals. In order to work with algebraic functions and their integrals, Riemann used the concept of surface (now called *Riemann surface*) of an algebraic function. Riemann's ideas on algebraic topology once more seem to have had a general influence on Betti's most famous work: *Sopra gli spazi di un numero qualunque di dimensioni* (1871) (which contains the so–called *Betti's numbers*) even if Riemann's texts on the *Analysis situs* had not yet been published. In 1868, however, Riemann's famous work for his *Habilitation* of 1854 had already come out: *Über die Hypothesen, welche der Geometrie zu Grunde liegen*[2].

Riemann's methods and ideas were closely linked to theoretical physics, which represented not only a field of research but also a source of inspiration for pure mathematics, so Betti directed his research also towards mathematical physics. The first work in 1860 on the propagation of plane waves took up the memoir by Riemann of the same year and later his interest in mathematical physics proceeded with a treatise on Newtonian forces published in 1879. Betti transferred the methods that Gauss and Green had applied to the integration of Laplace's equation, to the theories of elasticity and heat. His results opened up a field of research in Italy on the integration of equations of elasticity developed by Cerruti, Somigliana, and, more generally, became an important line of studies of mathematical physics to which Marcolongo, Tedone, Almansi, Lauricella, and Levi–Civita contributed[3].

Riemann's influence on Casorati was also important but in a different way[4]. From 1859 on Casorati held the Chair of Introduction to sublime calculus and firmly believed that research in the field of the theory of complex variables constituted the new frontier of mathematics. In accordance with the image of mathematical analysis of his day, which he described in an inaugural lecture to his course (17th January 1864), published only in 1997, are the research works of complex analysis and in particular the results of the German authors, his contemporaries, which developed *a current of ideas, destined ... to*

[1] U. Bottazzini, *Riemanns Einfluss auf E. Betti und F. Casorati* in: *Archive for History of Exact Sciences* 18, 1977, pp. 27–37, U. Bottazzini, *Va pensiero. Immagini della matematica nell'Italia dell'Ottocento*, Il Mulino, Bologna 1994, U. Bottazzini, *Il flauto di Hilbert. Storia della matematica*, Utet, Torino 2000.

[2] Publ. by R. Dedekind, after Riemann's death, in: *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13, 1868, pp. 134–150.

[3] V. Volterra, *Le matematiche in Italia nella seconda metà del secolo XIX* in: *Atti del IV Congresso Internazionale dei Matematici*, vol. 1, Roma 1908, pp. 53–65.

[4] V. Volterra, *Betti, Brioschi, Casorati, trois analystes Italiens et trois manières d'envisager les questions d'analyse* in: *Compte Rendu du deuxième Congrès International des Mathématiciens*, Paris 1902, pp. 43–57.

dominate, to open *a boundlessly rich region which is more or less unexplored*[1]. Within the European scientific production which he described, only the German school was able to provide the stimulus for new research with general principles of vast influence. Although not completely acceptable today, Casorati's description is representative of the opinions that Italian mathematicians had formed on the new research front in Europe.

The scene had been set by Cauchy and the extraordinary results in the theory of the elliptic functions of Abel and Jacobi in the years 1828–1829. In Jacobi's branch of studies there had appeared a second group of works around the 1850s, by Göpel (1849), Rosenhain (1851), Hermite (1855), Weierstrass (1856), and Riemann (1857). It was to the school of Riemann and his predecessor at Göttingen Dirichlet that Casorati attributed the merit of a definition of analytic functions based on conditions of continuity. Weierstrass and Riemann were the mathematicians who were really committed to developing new theories and Germany, as Casorati said, *brought again to the splendour of Kepler and Leibniz*[2].

France had contributed to the new course with the research works of Hermite on the transformation of Abelian functions and the applications to the resolution of fifth grade equations by means of elliptic functions (Liouville, Puiseux, Briot and Bouquet). After being the centre from which mathematical research radiated at the beginning of the century, France was now going through a period of decline: *at present there are few distinguished scholars of pure mathematics*[3]. No adequate school had been developed, the treatises themselves had hardly changed and being restricted to official programmes, unlike Germany, they all appeared the same, aspiring, not to new ideas, but rather to equip the student with practical applications through the simplest and most rapid means possible.

England possessed a number of analysists of great worth: Cayley, Sylvester, Boole, Hamilton, and Salmon, but apart from a few works of secondary interest by Cayley, the English line of research in mathematics was mostly concerned with the theory of forms. Unlike Germany, in France and Italy there were no treatises on algebraic analysis which constituted an introduction to higher analysis. Even the recent treatise by Boole (1859), a good work of revision, dealt with differential equations using old methods. English books, in Casorati's opinion, isolated mathematical activity from that of the Continent just as had previously happened after the famous Newton – Leibniz quarrel. They were developing some aspects extensively and with great clarity, while completely neglecting others.

In Italy the disproportion was even more marked: before 1858 only a few works on elliptic functions were reported in the *Annali* and after the reform of Brioschi there appeared a translation of Riemann's memoir on the theory of

[1] A. Capelo, M. Ferrari & P. Moglia, *Un discorso di Felice Casorati sull'analisi matematica del suo tempo* in: *L'insegnamento della matematica e delle scienze integrate* 20 B, 3/1997, pp. 209–266.

[2] A. Capelo, M. Ferrari & P. Moglia, *Un discorso di Felice Casorati ...* , p. 229.

[3] A. Capelo, M. Ferrari & P. Moglia, *Un discorso di Felice Casorati ...* , p. 234.

Abelian functions (*Crelle's Journal* 54, 1857) and Betti's monograph on elliptic functions. Only Enrico Betti, after his brilliant success of the resolution of algebraic equations, had given some important contributions to the new field of research with his memoirs on the algebraic functions of a complex variable.

The school of mathematics at St. Petersburg presented only two memoirs by Tchebichef on the new line of research on integral calculus, in which he resolved the problem of integration of differentials containing a square root of third and fourth degree polynomials. However, the solution was extremely complicated and did not allow further developments, since *it must be considered more as a monument to the fine brain of its author rather than a further step on the ladder of progress*[1].

In his description of contemporary mathematicians, with the exception of Enrico Betti, the authors emphasized by Casorati were all German: Weierstrass, Riemann, Roch, Schlömilch, and Stern for pure mathematics, Borchardt and Kronecker for applied mathematics. Casorati pointed out the modernity of these authors in comparison with other less innovative works linked to an outdated geometric foundation, *which always claimed to rebuild the construction with new forms of a general geometric nature*[2]. Among these latter authors he listed Bjerknes, Mourey, Marie, and Clayeux and as for Bellavitis's theory of equipollence, Möbius's barycentric calculus, and Hamilton's theory of quaternions he did not see a bright future.

In 1868 Casorati published the *Teorica delle funzioni di variabili complesse*, which was also the text for his lessons at the University of Pavia[3]. This book, in which the ideas and techniques of Cauchy, Weierstrass and Riemann are joined, greatly contributed to the dissemination, particularly in Italy, of the theory of functions of a complex variable, of which elliptic functions constituted a part.

Casorati also carried forward interesting research work on functions with multiple periods, following one of Goepel's criticisms of Jacobi's theorem on periodicity of Abelian functions of one variable. Casorati discussed his results with Weierstrass in 1864 shortly after the publication in *Comptes Rendus*.

4. The long wave of German influence

The influence of the German schools extended to the second generation of Italian mathematicians and their pupils. For this paper, the most celebrated and representative have been chosen: Eugenio Beltrami, Ulisse Dini, and Giuseppe Peano[4].

A pupil of Brioschi at Pavia, Beltrami (1835–1900) became temporary professor of Algebra and Analytical Geometry in 1862 at the University of

[1] A. Capelo, M. Ferrari & P. Moglia, *Un discorso di Felice Casorati ...* , p. 239.

[2] A. Capelo, M. Ferrari & P. Moglia, *Un discorso di Felice Casorati ...* , p. 232.

[3] First Volume, Fratelli Fusi, Pavia 1868. The second volume was never published.

[4] *Opere matematiche di Eugenio Beltrami*, Hoepli, Milano 1902–1920, 4 vols, U. Dini, *Opere*, UMI, Cremonese, Roma 1953–1959, 5 vols, G. Peano, *Opere scelte*, UMI, Cremonese, Roma 1957–1959, 3 vols.

Bologna and then, in 1864, he obtained the chair of Geodesy at the University of Pisa, where he formed a friendship with Enrico Betti and met Bernhard Riemann. Beltrami devoted himself to differential geometry, following on from the works of Lobachevsky, Gauss, Riemann and Luigi Cremona. He translated into Italian the work by Gauss on conformal representation and he studied the problem of establishing when it is possible to represent a geodetic of a surface by means of a rectilinear segment on the plane: he discovered that it was possible only for surfaces with constant curvature. Riemann's *Habilitationsschrift*, regarding the hypotheses which form the basis of geometry, deeply influenced Beltrami, who, in 1868 released his most famous work: the *Saggio sopra un'interpretazione della geometria non euclidea*, in which the hyperbolic plane is represented inside a real disk. In this article Beltrami also provided a model of Lobachevskian geometry on a surface with negative constant curvature (the pseudosphere, a surface generated by rotating a tractrix around its asymptote). In 1869 in his work entitled *Teoria fondamentale degli spazi a curvatura costante*, he went into greater depth on the question by demonstrating that Lobachevskian geometry coincided with that of a surface with negative constant curvature and provided, for the first time, a treatment of Riemann's spherical geometry in the language of differential geometry. Beltrami's model, later studied also by Felix Klein for the metric properties, greatly contributed to the dissemination of Non–Euclidean geometries[1].

Together with Cremona and Beltrami, Ulisse Dini (1845–1918) was one of the leading Italian mathematicians in the late 19th century. His extremely original contributions were part of the research work that led to the critical revision of the foundations of real analysis as well as the concept of integral. His scientific publications range from the theory of the functions of real variables, the theory of sets, differential geometry of surfaces, to the theory of differential equations and analytical functions. Dini was also politically active, he held the post of Deputy and Senator and was, moreover, on several occasions Vice President of the Upper Council of the Ministry of Education. He studied under Betti and Mossotti at the *Scuola Normale Superiore* in Pisa, where he graduated in 1864 and began to publish his works on differential geometry. The following year he specialized in Paris under Charles Hermite and Joseph Bertrand. In 1866 he returned to Pisa, where he was given a teaching post of geodesy and higher algebra. In 1871, Betti left his chair in analysis and higher geometry preferring to take up physical mathematics. Besides research work, Dini devoted himself to a critical revision of the foundations of infinitesimal calculus, which he tested during his lessons. In 1877 he was also given a chair in infinitesimal calculus and published a famous treatise on the foundations of the theory of functions of real variables[2]

[1] L. Boi, L. Giacardi & R. Tazzioli, *La découverte de la géométrie non euclidienne sur la pseudosphère. Les lettres de Beltrami à Hoüel, avec une introduction des notes et commentaires critiques*, Blanchard, Paris 1998, R. Tazzioli, *Beltrami e i matematici 'relativisti'. La meccanica in spazi curvi nella seconda metà dell'Ottocento*, Quaderni dell'U.M.I. n. 47, Pitagora, Bologna 2000.

[2] U. Dini, *Fondamenti per la teorica delle funzioni di variabili reali*, T. Nistri, Pisa 1878. A book on Fourier series and analytic expansions came out two years later: U. Dini, *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*, Nistri, Pisa 1880.

which collected the results presented in his university courses and became the first systematic exposition of the new ideas that were being outlined in works by Weierstrass, Heine, Hankel, Du Bois–Reymond, Schwarz, Dedekind, Cantor and Riemann. Dini's *Fondamenti* were translated into German by J. Luroth and A. Schepp and then, even 14 years after the Italian edition, as the editors stated, it was still considered to be *the only book of modern theory of a real variable*[1].

The volume commenced with Dedekind's theory on real numbers and the *gruppi di punti* (*Punkmenge* of Cantor), namely the concepts of sequence of numbers, limit point, derived set, set of first and second category, least upper bound and greatest lower bound. There then followed the theory of the limits of sequences posed on rigorous bases, continuous and various types of discontinuous functions, uniform continuity (with Cantor's theorem as demonstrated by Schwarz), Hankel's concept of pointwise discontinuous and totally discontinuous functions, derivatives and derivability, intermediate value theorem (correctly demonstrated), types of discontinuity of the derivate function, series ad series of functions, uniform convergence and its applications to continuity, differentiability and integrability, Hankel's principle of condensation of singularities, continuous everywhere but nowhere derivable functions, functions of bounded variation, oscillation of a function, etc.

Following this there was a section on finite integration which merits a separate description[2]. The condition for integrability of a bounded function over an interval, provided by Riemann in another famous memoir *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*, presented in 1854 as his *Habilitationsschrift*, had extended the definition of integral also to functions that have a dense set of points of discontinuity. There followed several works in Germany, Italy, France and England, destined to clarify and extend Riemann's ideas on integration. Hankel had conferred a characterization to the discontinuities of a function and had deduced some consequences – in part erroneous – on integrability. Hankel's condition was to remain a focus of attention for several years: among the most important works on Riemann integrability that were later published we should remember those by Darboux, Ascoli and Smith. In particular, Giulio Ascoli, a Professor at the *Istituto Tecnico Superiore* (Polytechnic) in Milan, published in 1875 a work which gave a demonstration of a necessary and sufficient condition equivalent to Riemann integrability, which sees the intervention not of oscillation intended as an interval function, but as a point function. In his *Fondamenti* Dini took up again and corrected Hankel's memoir, by introducing the study

[1] A second volume on the functions of several variables was planned, but it was not published. The functions of several variables, however, with the theory of maxima and minima and Dini's famous theorem of implicit functions, length of curves and surface areas, differential equations and the other topics that make up a modern course on analysis were inserted into the complete course: U. Dini, *Lezioni di Analisi infinitesimale*, Nistri, Pisa, *volume 1: Calcolo Differenziale* (1907), *Volume II. Calcolo Integrale* (1915).

[2] See P. Nalli, *Esposizione e confronto critico delle diverse definizioni proposte per l'integrale definito di una funzione limitata o no*, reprinted in: P. Nalli, *Opere scelte*, U.M.I, Vizzi, Palermo 1976, G. Letta, *Le condizioni di Riemann per l'integrabilità e il loro influsso sulla nascita del concetto di misura* in: *Rendiconti dell'Accademia Nazionale dei XL* 18, 1/1994, pp. 143–169.

of *punteggiate discontinue* as he called discontinuous functions on a nowhere dense set of points considered by Hankel. Dini's work was also noticeably influenced by Cantor's ideas on point sets and particularly on the use of the topological notion of derived set and set of first category. While Dini was quite clear on the difference between meagre set, nowhere dense set and negligible set, some confusion may be found in the works of his contemporaries, Du Bois–Reymond and Harnack. Cantor, himself, in the first of a long series of articles on transfinite numbers[1] and in a successive work in 1882[2] on a new general principle of the condensation of singularities of functions, gave Dini the credit for having developed the concept of derived set by placing it at the basis of important generalisations of well–known analytical principles, as well as having clarified Hankel's condition. Dini introduced interesting criteria of integrability, such as the one which states that the set of discontinuity points forms a negligible set. Continuing this line of research, one of his pupils, Vito Volterra who was to become famous, in a work of 1881, presented an example of a nowhere dense not negligible set to complete Dini's discussion as well as new criteria of integrability through use of jump function.

The much celebrated Giuseppe Peano (1858–1932) belongs to the generation after Brioschi, Betti and Casorati, but his formation took place outside the schools of Pisa and Pavia[3]. When comparing his position on the foundations of real analysis with that of his mentor, Genocchi, the influence of the German school clearly emerges (Weierstrass, Du Bois Raymond, Dedekind). The relationship between research and teaching was strengthened in the second half of the 19th century since studies on the foundations of mathematics affected the structure of elementary and higher treatises.

Genocchi, professor of integral and differential calculus at the University of Turin, had himself renewed the teaching of calculus in the light of modern research. The influence of the French school was naturally stronger in Turin than elsewhere and Genocchi, who was open to European culture particularly through his epistolary contacts with the most important mathematicians of the day, had introduced innovations to his course which was greatly influenced by Cauchy's *Cours d'Analyse* and other texts by C. Hermite, E. Hoppe, and J. Hoüel. If this was considered progress compared to the previous situation, which was dominated by the Lagrangian tradition making it difficult to introduce Cauchy's methods, further renewal may be seen to be brought in by Giuseppe Peano, Genocchi's assistant, who substituted him in the teaching of infinitesimal analysis over the years 1882–1883 when Genocchi was taken ill. Linked to these lessons was the famous counterexample found by Peano in

---

[1] G. Cantor, *Über unendliche, lineare Punktmannichfaltigkeiten* in: *Mathematische Annalen* 15, 1879, p. 1.

[2] G. Cantor, *Über ein neues und Allgemeines Kondensationsprinzip der Singularitäten von Funktionen* in: *Matematische Annalen* 19, 1882, pp. 588–594.

[3] H. C. Kennedy, *Life and Works of Giuseppe Peano*, Reidel, Dordrecht 1980.

concomitance with Schwarz, which demonstrated the inadequacy of the definition of curved surface area provided by Serret[1].

Thus there arose a treatise with very particular features, rich in historical and critical notes in which commonly accepted mistakes were corrected and counterexamples presented which were to become classic[2]. The work was destined to become a reference point for successive treatises which Pringsheim inserted among the best textbooks on calculus[3]. Among the issues clarified as far as the foundations of mathematical analysis were concerned, we may mention the many counterexamples on the convergence of numerical series, the properties of functions continuous or derivable over an interval, the formula of the remainder in a series expansion, the exchange of order of differentiation in mixed partial derivatives[4].

The structure of the course was changed and the more specifically algebraic contents were eliminated to leave more space to topics typical of mathematical analysis, the theory of limits of real variables, continuous functions, derivates, series, series expansions and series of functions, functions of several variables, maxima and minima, functions of complex variables, indefinite and definite integrals. Among the modern authors quoted we find: Dedekind, Cantor, Heine, Harnack, Hoppe, Lipschitz, Du Bois–Raymond, Lejeune–Dirichlet, Jacobi, Pasch, Pringsheim, Weierstrass, Riemann, Schwarz, Schlömilch, Stolz, Wiener, and Baltzer. Among those of French nationality we find Cauchy, Darboux, Bouquet, Hermite, Hoüel, Jordan, Terquem, and Serret, but the last mentioned being especially criticised by exposing examples of his inexactness, thus giving rise to controversy such as that with Philippe Gilbert. Criticism was also directed at Hermite and Sturm. Among the Italians Ulisse Dini was particularly quoted, but also Volterra, Tardy, Fergola, Faà di Bruno, Bellavitis, and as for the English, Cayley and Thodunther.

Research on the foundations of analysis continued with the successive treatise called *Applicazioni geometriche del calcolo infinitesimale* (1887) in which, among other things, a new definition of measure and measurable set was introduced, which changed the *Inhalt* theory of the German authors (Harnack, Du Bois–Reymod, Stolz, Cantor) and which would then also be developed by Jordan. Moreover, the functions of sets were studied and the theory of quaternions was explained. An aspect of Peano's modernity concerned the use of the concept of the least upper bound (*limite superiore*): this concept originates from the research carried out by Weierstrass on the

---

[1] See M. T. Borgato, *Giuseppe Peano: tra analisi e geometria* in: *Peano e i fondamenti della matematica*, Accademia Nazionale di Scienze Lettere e Arti di Modena, Modena 1992, pp. 139–169.

[2] A. Genocchi, *Calcolo differenziale e principii di calcolo integrale, pubblicato con aggiunte dal Dr Giuseppe Peano*, Bocca, Torino 1884.

[3] See A. Pringsheim, *Grundlagen der allgemeinen Funktionenlehre* in: *Encyklopädie der mathematischen Wissenschaften*, Bd. 2: *Analysis*, Teubner, Leipzig 1899–1904, II A 1, pp. 1–53.

[4] See Peano's letters to Genocchi and Jordan related to these topics in: M. T. Borgato, *Alcune lettere inedite di Peano a Genocchi e a Jordan sui fondamenti dell'analisi* in: *Angelo Genocchi e i suoi interlocutori scientifici*, (eds) A. Conte & L. Giacardi, Centro Studi per la Storia dell'Università di Torino, Torino 1991, pp. 61–97.

properties of continuity of real numbers, whose construction, as is well–known, starting from rational numbers, was realized by various authors with different constructions around the year 1872. Peano was the first to use the concept of least upper bound (or supremum) as the pivot for every key definition in the theory of functions of real variables: in the definition of measure of a set, length of a curve or integral of a function. However the French mathematician Camille Jordan, with whom Peano shares merit for introducing the homonymous measure theory, continued to use the concept of limit, which always requires a demonstration of its existence, at least to justify that definition.

In 1888 the influence of Grassmann's work is evident and declared in the title of the new treatise of *Calcolo geometrico*[1], in which a calculus of geometric entities was constructed analogous to the algebraic one and in the language of the *Ausdehnungslehre* the fundamental results of infinitesimal calculus were translated. The first part explains the operations of deductive logic which mark the beginning of Peano's research on the application of mathematical logic (reference to Hamilton, Cayley, Boole and Grassmann). 1890 saw the translation into the language of the bivettors of his definition of the area of a curved surface. In 1889 there was the publication of his celebrated *Arithmetices Principia* which contained the axiomatisation of natural numbers (Peano Axioms), the credit for which Peano shared with Dedekind[2]. Within the context of topological questions of the real line originated by the German school (Weierstrass, Cantor in particular) may be placed the most celebrated product of Peano's research in this field, the famous Peano's curve, the first example of space–filling curve which is crucial to the theory of dimension – developed by Cantor and Eugen Netto – rectificability and quadrature[3].

In conclusion, the first forty years after the Unification brought Italian mathematics to the highest levels in the world. To crown this success the 4th Congress of Mathematics held in 1908 in Rome secured international recognition, being second only to those held in Zurich (1897), Paris (1900) and Heidleberg (1904). The best achievements had been gained between the two centuries by Volterra and Arzelà in analysis, Castelnuovo, Enriques, and Severi in geometry, but the first most difficult steps were accomplished by Betti, Brioschi, Cremona, Casorati, and Beltrami during the first decade of United Italy[4].

---

[1] G. Peano, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann*, Bocca, Torino 1888.

[2] G. Peano, *Arithmetices Principia, nova methodo exposita*, Bocca, Torino 1889.

[3] G. Peano, *Sur une curve qui replit toute une aire plane* in: *Mathematische Annalen* 36, 1890, pp. 157–160. On Peano see also E. Luciano, *Peano docente e ricercatore di analisi 1881–1919*, Ph.D. Thesis, Università di Torino 2008 and E. Luciano & S. C. Roero, *G. Peano matematico e maestro*, Università di Torino, Torino 2008.

[4] A. Guerraggio & P. Nastasi, *Roma 1908: il Congresso Internazionale dei Matematici*, Bollati Boringhieri, Torino 2008.